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A different approach to prove the Baker-Campbell-Hausdorff (BCH) formula

To prove the Baker-Campbell-Hausdorff formula for a linear operator on Hilbert space

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \sum_{k=0}^{\infty} \frac{1}{k!}[\hat{A},\hat{B}]_k \tag{1}$$

Lemma: $[\hat{A},\hat{B}]_n = [\hat{A},[\hat{A},\hat{B}]_{n-1}] = \sum_{i=0}^{i=n}(-1)^i\binom{n}{i}\hat{A}^{n-i}\hat{B}\hat{A}^i$

Proof by induction:

For the sake of brevity, we will drop hats over A and B. For $n = 1$, we get, $[A, B]_1 = AB - BA$

This is too trivial, so let's look at $n = 2$ and make that our base case.

For $n = 2$, we get, $[A, B]_2 = [A, [A, B]_1] = [A, (AB - BA)] = A^2B - 2ABA - BA^2$

The above expression for $[A, B]_2$ can also be verified by our claim.

Let's assume our induction hypothesis (claim) holds for $n = k$,

Thus we have $[A, B]_k = \sum_{i=0}^{i=k}(-1)^i\binom{k}{i}A^{k-i}BA^i$

Now, we show that the hypothesis holds for $n = k + 1$,

Thus, $[A, B]_{k+1} = [A, [A, B]_k] = A[A, B]_k - [A, B]_kA$, must hold. Evaluating the right hand side, we get,

$$= A\sum_{i=0}^{i=k}(-1)^iA^{k-i}BA^i\binom{k}{i} - \left(\sum_{i=0}^{i=k}(-1)^iA^{k-i}BA^i\binom{k}{i}\right)A$$
$$= \sum_{i=0}^{i=k}(-1)^iA^{k+1-i}BA^i\binom{k}{i} + \sum_{i=0}^{i=k}(-1)^{i+1}A^{k-i}BA^{i+1}\binom{k}{i}$$

now, let $i + 1 = p$ in the second sum, so, the limits change to $p = 1$ to $p = k + 1$. Also, let's evaluate $i = 0$ case and $p = k + 1$ in the first and second sum, respectively

for $i = 0$, the term is, $\binom{k}{0}A^{k+1}B = A^{k+1}B$

for $p = k + 1$, the term is, $(-1)^{k+1}\binom{k}{k}BA^{k+1}$

Hence, the final expression becomes,

$$= A^{k+1}B + \sum_{i=1}^{i=k}(-1)^i\binom{k}{i}A^{k+1-i}BA^i + \sum_{p=1}^{i=k}(-1)^p\binom{k}{p-1}A^{k+1-p}BA^p + (-1)^{k+1}BA^{k+1}$$

Now, the i and p sums could be combined, as the *form* is same; giving,

$$= A^{k+1}B + (-1)^{k+1}BA^{k+1} + \sum_{i=1}^{i=k}(-1)^i\binom{k+1}{i}A^{k+1-i}BA^i$$

Where we used Pascal's rule,

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

Now, notice that the two terms which are outside the sum can be incorporated into the sum by changing limits on i from 0 to $k+1$. Hence, we get,

$$= \sum_{i=0}^{i=k+1} (-1)^i \binom{k+1}{i} A^{k+1-i} B A^i$$

Therefore, by the principle of mathematical induction, our claim must hold for all $n \in \mathbb{N}$.

Now, let's evaluate the right hand side of equation-1.

$$\begin{aligned} &= \sum_{k=0}^{k=\infty} \frac{1}{k!} [A, B]_k \\ &= \sum_{k=0}^{k=\infty} \frac{1}{k!} \sum_{i=0}^{i=k} (-1)^i \frac{k!}{(k-i)!\ i!} A^{k-i} B A^i \end{aligned}$$

$\frac{1}{k!}$ gets in the sum, as the sum doesn't depend on k explicitly, and which cancels with $k!$ in the numerator. Let's introduce another variable j as, $j = k - i$. We can now write the final expression to be,

$$= \sum_{k=0}^{k=\infty} \left(\sum_{i=0}^{i=k} \left(\sum_{j=0}^{j=k} \frac{A}{j!} \right) B \left\{ (-1)^i \frac{A^i}{i!} \right\} \right)$$

Now with both sums running up to infinity for both the sums with indices, i and j. The terms in the brackets are precisely the Taylor series expansion (in the neighbourhood of $\vec{0}$) for exponential of an operator with $\hat{B}$ *sandwiched* between the two $\hat{A}$ exponentials.

Therefore the right hand side of equation-1 evaluates to $\boxed{e^{\hat{A}}\, \hat{B}\, e^{-\hat{A}}}$; which is the left hand side of equation-1.

This proves equation-1. (Remember, LHS = RHS $\iff$ RHS = LHS)